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ON THE UNCOLOURED QUANTUM LINK INVARIANTS ARISING FROM $U_q(\mathfrak{osp}(1|2n))$ AND $U_{-q}(\mathfrak{so}(2n+1))$ *

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ABSTRACT

The $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ quantum link invariants obtained by colouring each component of a link with the $(2n+1)$ -dimensional irreducible representations of each quantum algebra, respectively, are the same up to a possibly obscure substitution of variables.

These quantum link invariants are identical to the Kauffman link invariants obtained from two isomorphic specialisations of the Birman-Wenzl-Murakami algebra. Each of these Kauffman link invariants is the trace of some element in the relevant specialisation of the Birman-Wenzl-Murakami algebra.

The truncated Bratteli diagrams of the semisimple quotients of these two specialisations of the Birman-Wenzl-Murakami algebra are the same, where the quotients are taken with respect to an ideal generated by a subset of elements with zero trace.

The trace of any element in one specialisation of the Birman-Wenzl-Murakami algebra is the same as the trace of the corresponding element in the other specialisation of the Birman-Wenzl-Murakami algebra up to a possibly obscure substitution of variables. This gives the corresponding result for the relevant uncoloured quantum link invariants.

Keywords: link invariants, quantum algebra, Birman-Wenzl-Murakami algebra

Mathematics Subject Classification 2000: 57M27, 17B37

1. Introduction

The essential result of this paper is as follows: let L be a link and $F_L(r, s)$ its associated Kauffman link invariant where $r, s \in \mathbb{C}$ are parameters in the Birman-Wenzl-Murakami algebra $BMW_f(r, s)$ [6]. Let r' and s' be complex parameters such that the Bratteli diagrams for the semisimple quotients of $BMW_f(r, s)$ and $BMW_f(r', s')$, obtained by taking the quotient with respect to an annihilator ideal generated by a subset of elements with zero trace, are identical. Then $F_L(r, s)$ and $F_L(r', s')$ are identical up to a substitution of variables.

This substitution may be difficult to do in practise if r is a function of s , as further knowledge about the explicit form of the trace of a certain element in

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BMW_f , the unspecialised form of the Birman-Wenzl-Murakami algebra, may be needed. Such a substitution, however, exists in principle.

We illustrate this for $(r, s) = (-q^{2n}, q)$ and $(r', s') = (q^{2n}, -q)$, where q is a non-zero generic complex number (ie not a root of unity), the Kauffman link invariants for which are identical to the quantum link invariants obtained by colouring each component of the link L with the $(2n + 1)$ -dimensional irreducible representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n + 1))$, respectively.

This paper draws on representations of two specialisations of the Birman-Wenzl-Murakami algebra that are obtained from representations of the Braid group defined on repeated tensor products of the $(2n + 1)$ -dimensional irreducible (vector) representations of $U_{-q}(\mathfrak{so}(2n + 1))$ and $U_q(\mathfrak{osp}(1|2n))$ [2,5].

This paper was motivated not by a comparison of the relevant Kauffman or quantum link invariants, but by an observation that the structures of certain semisimple quotients of two specialisations of the Birman-Wenzl-Murakami algebra are identical, and thus the relevant Kauffman link invariants (and uncoloured quantum link invariants) have a particular symmetry that may not be evident upon first inspection.

2. Link invariants from $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n + 1))$

2.1. Link invariants from Markov traces

Let us fix some notation: we fix \mathbb{Z} to be the set of integers and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

In this subsection we recall how one can obtain a link invariant from a Markov trace [7]. This material is well-known, but we recall it here to keep this paper self-contained. In Subsec. 2.2 we recall relevant data about the quantum algebras $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n + 1))$ and their representations and in Subsec. 2.3 we describe certain Markov traces on those representations.

Let B_f be the braid group on f strings with generators $\{\sigma_i^{\pm 1} \mid i = 1, \dots, f - 1\}$ as shown in Fig. 1.

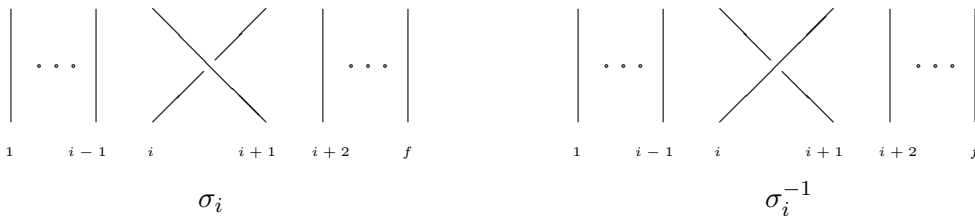


Fig. 1. The generators σ_i and σ_i^{-1} of the Braid group B_f .

Let a link L be presented as the canonical closure of a braid on f strings with

corresponding braid group element $b = \sigma_{i_1}^{m_1} \sigma_{i_2}^{m_2} \cdots \sigma_{i_j}^{m_j} \in B_f$, where B_f is the braid group on f strings and where $m_k \in \mathbb{Z}$ for each k , and let ρ be a nontrivial representation of B_f . Then given a functional $\psi : \rho(B_f) \rightarrow \mathbb{C}$ satisfying the three Markov properties:

- (i) $\psi(\rho(\theta_i \theta_j)) = \psi(\rho(\theta_j \theta_i)), \forall \theta_i, \theta_j \in B_f$,
- (ii) $\psi(\rho(\theta \sigma_{f-1})) = z\psi(\rho(\theta)), \forall \theta \in B_{f-1} \subset B_f$,
- (iii) $\psi(\rho(\theta \sigma_{f-1}^{-1})) = \tilde{z}\psi(\rho(\theta)), \forall \theta \in B_{f-1} \subset B_f$,

where we take ψ on the right hand sides of Eqs. (ii) and (iii) to be defined on $\rho(B_{f-1})$ where B_{f-1} is generated by $\sigma_i^{\pm 1}$ for $i = 1, 2, \dots, f-2$, one can define a link polynomial:

$$\tilde{F}(L) = (z\tilde{z})^{-(f-1)/2} (\tilde{z}/z)^{e(b)/2} \psi(\rho(b)), \quad (2.1)$$

where $e(b) = \sum_{k=1}^j m_k$.

2.2. Representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$

To discuss Markov traces arising from representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$, we need to discuss the algebras and elements of their representation theory which we do to keep this paper self-contained [8].

Let H^* be an n -dimensional complex vector space with a basis $\{\epsilon_i | i = 1, 2, \dots, n\}$ and let $(\cdot, \cdot) : H^* \rightarrow \mathbb{C}$ be a bilinear form given by $(\epsilon_i, \epsilon_j) = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Let $\{\alpha_i | i = 1, 2, \dots, n\}$ be a basis of simple roots of H^* : we fix $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \leq n-1$ and $\alpha_n = \epsilon_n$. The Cartan matrices of the Lie superalgebra $\mathfrak{osp}(1|2n)$ and the Lie algebra $\mathfrak{so}(2n+1)$ are the same and are given by $A = (A_{ij})_{i,j=1}^n$ where $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

The set of positive roots is $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$ where $\Phi_0^+ = \{\epsilon_i \pm \epsilon_j, 2\epsilon_k | 1 \leq i < j \leq n, 1 \leq k \leq n\}$ is the set of positive even roots and $\Phi_1^+ = \{\epsilon_k | 1 \leq k \leq n\}$ is the set of positive odd roots.

Let q be a non-zero complex number satisfying $q^2 \neq 1$. The Jimbo version of the quantum superalgebra $U_q(\mathfrak{osp}(1|2n))$ is a \mathbb{Z}_2 -graded Hopf algebra generated by $\{e_i, f_i, k_i^{\pm 1} | i = 1, 2, \dots, n\}$ where all the generators are graded to be even except for e_n and f_n , which are graded to be odd. The generators are subject to the following relations:

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \quad i < n, \quad e_n f_n + f_n e_n = \frac{k_n - k_n^{-1}}{q - q^{-1}},$$

$$k_i e_j k_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \quad k_i f_j k_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j, \quad i, j \leq n, \quad (2.2)$$

$$k_i^{\pm 1} k_j^{\pm 1} = k_j^{\pm 1} k_i^{\pm 1}, \quad k_i^{\pm 1} k_j^{\mp 1} = k_j^{\mp 1} k_i^{\pm 1}, \quad i, j \leq n, \quad (2.3)$$

together with the quantum Serre relations which we will not be directly using in this paper and which can be found in [8].

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In dealing with representations of the Braid group, we will use the co-algebra structure of $U_q(\mathfrak{osp}(1|2n))$. The co-multiplication is an algebra homomorphism $\Delta : U_q(\mathfrak{osp}(1|2n)) \rightarrow U_q(\mathfrak{osp}(1|2n)) \otimes U_q(\mathfrak{osp}(1|2n))$ defined by

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$

for $i = 1, 2, \dots, n$, and the co-unit $\epsilon : U_q(\mathfrak{osp}(1|2n)) \rightarrow \mathbb{C}$ is a homomorphism defined by

$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad \epsilon(k_i^{\pm 1}) = \epsilon(1) = 1, \quad i = 1, 2, \dots, n.$$

If an element $x \in U_q(\mathfrak{osp}(1|2n))$ has a grading, we indicate its grading by writing $[x] = 0$ if x is even and $[x] = 1$ if x is odd.

If each of two elements $x, y \in U_q(\mathfrak{osp}(1|2n))$ has a grading, then the element $xy \in U_q(\mathfrak{osp}(1|2n))$ also has a grading given by $[xy] = ([x] + [y]) \pmod{2}$. In addition, the element $x \otimes y$ has a grading given by $[x \otimes y] = ([x] + [y]) \pmod{2}$.

The quantum superalgebra $U_q(\mathfrak{osp}(1|2n))$ has a \mathbb{Z}_2 -grading:

$$U_q(\mathfrak{osp}(1|2n)) = U_q(\mathfrak{osp}(1|2n))_0 \oplus U_q(\mathfrak{osp}(1|2n))_1$$

where $U_q(\mathfrak{osp}(1|2n))_i = \{x \in U_q(\mathfrak{osp}(1|2n)) \mid [x] = i\}$. We say that an element $x \in U_q(\mathfrak{osp}(1|2n))$ is homogeneous if $x \in \bigcup_{i=0}^1 U_q(\mathfrak{osp}(1|2n))_i$.

We introduce the graded permutation operator

$$P : U_q(\mathfrak{osp}(1|2n)) \otimes U_q(\mathfrak{osp}(1|2n)) \rightarrow U_q(\mathfrak{osp}(1|2n)) \otimes U_q(\mathfrak{osp}(1|2n))$$

which acts on homogeneous elements $x, y \in U_q(\mathfrak{osp}(1|2n))$ as

$$P(x \otimes y) = (-1)^{[y][x]} y \otimes x,$$

the action of which is extended to inhomogeneous elements by linearity.

We also mention that $U_q(\mathfrak{osp}(1|2n)) \otimes U_q(\mathfrak{osp}(1|2n))$ is a \mathbb{Z}_2 -graded algebra with multiplication

$$(a \otimes b)(x \otimes y) = (-1)^{[b][x]} ax \otimes by,$$

for homogeneous elements $a, b, x, y \in U_q(\mathfrak{osp}(1|2n))$, which extends to inhomogeneous elements by linearity.

Let $\bar{\pi}$ be any representation of $U_q(\mathfrak{osp}(1|2n))$ with corresponding module W . Given an element $X \in \text{End}_{\mathbb{C}}(W)$, the quantum supertrace of X is defined to be

$$\text{str}_q(X) = \text{str}(\bar{\pi}(k_{2\rho}) \circ X),$$

where str is the usual supertrace and $k_{2\rho}$ is a product of the k_i 's such that $k_{2\rho} e_i k_{2\rho}^{-1} = q^{(2\rho, \alpha_i)} e_i$ for all i , where $2\rho \in H^*$ is defined by

$$2\rho = \sum_{\alpha \in \Phi_0^+} \alpha - \sum_{\beta \in \Phi_1^+} \beta = \sum_{i=1}^n (2n - 2i + 1) \epsilon_i.$$

The quantum superdimension of $\bar{\pi}$ is defined to be the quantum supertrace of the identity map on W : $\text{sdim}_q(\bar{\pi}) = \text{str}(\bar{\pi}(k_{2\rho}))$.

When q is not a root of unity, the finite dimensional irreducible highest weight representations of $U_q(\mathfrak{osp}(1|2n))$ that are deformations of irreducible representations of $\mathfrak{osp}(1|2n)$ are completely labelled by their highest weights. The set of the highest weights of all these irreducible representations of $U_q(\mathfrak{osp}(1|2n))$ is the set of all integral dominant weights $P^+ \subset H^*$ [8]. We say that an element $\lambda \in H^*$ is integral if it satisfies $l_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ for all $i < n$ and if $l_n = (\lambda, \alpha_n)/(\alpha_n, \alpha_n) \in \mathbb{Z}$ and that it is integral dominant if $l_i \in \mathbb{Z}_+$ for all $i = 1, 2, \dots, n$.

We label the finite dimensional irreducible representation of $U_q(\mathfrak{osp}(1|2n))$ with integral dominant highest weight λ by $\bar{\pi}_\lambda$ and denote its corresponding module by V_λ .

The quantum algebra $U_q(\mathfrak{so}(2n+1))$ is generated by $\{E_i, F_i, K_i^{\pm 1} \mid i = 1, 2, \dots, n\}$ subject to the relations

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad \forall i, j,$$

the relations (2.2) and (2.3) where we replace e_i, f_i and $k_i^{\pm 1}$ with E_i, F_i and $K_i^{\pm 1}$, respectively, and the quantum Serre relations which can be found in [8].

At generic q , each integral dominant weight λ is the highest weight of a finite dimensional irreducible $U_{-q}(\mathfrak{so}(2n+1))$ representation π_λ with the same dimension as the irreducible $U_q(\mathfrak{osp}(1|2n))$ representation $\bar{\pi}_\lambda$. (We also denote the module corresponding to π_λ by V_λ .) This result is a consequence of the isomorphism between $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ restricted to those representations of $U_q(\mathfrak{osp}(1|2n))$ in which all the k_i 's take integer powers of q as eigenvalues and restricted to the representations of $U_{-q}(\mathfrak{so}(2n+1))$ in which all the K_i 's take integer powers of $-q$ as eigenvalues [8].

Both $U_{-q}(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$ admit $(2n+1)$ -dimensional irreducible representations (sometimes called the vector representations) the modules of which we both denote by V . Whenever we consider V to be a $U_q(\mathfrak{osp}(1|2n))$ -module, we take the grading of its highest weight vector to be *odd*.

Let π be a representation of $U_q(\mathfrak{so}(2n+1))$ with corresponding module W and let $X \in \text{End}_{\mathbb{C}}(W)$. The quantum trace of X is defined to be $\text{tr}_q(X) = \text{tr}(\pi(K_{2\rho}) \circ X)$, where tr is the usual trace and $K_{2\rho}$ is a product of the K_i 's such that $K_{2\rho} E_i K_{2\rho}^{-1} = q^{(2\rho, \alpha_i)} E_i$ for all i . The quantum dimension of π is defined to be the quantum trace of the identity map on W : $\dim_q(\pi) = \text{tr}(\pi(K_{2\rho}))$. (Note that the 2ρ in $U_q(\mathfrak{so}(2n+1))$ is identical to the 2ρ in $U_q(\mathfrak{osp}(1|2n))$.)

2.3. Braid group representations and Markov traces from $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$

In this subsection we discuss representations of the Braid group arising from representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ and also describe Markov traces arising from these Braid group representations.

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It is well known that there exist invertible maps $R_{\lambda\mu}^{so(2n+1)} \in \text{End}_{\mathbb{C}}(V_{\lambda} \otimes V_{\mu})$ for all integral dominant weights λ and μ satisfying

$$R_{\lambda\mu}^{so(2n+1)} \cdot (\pi_{\lambda} \otimes \pi_{\mu})\Delta(x) = (\pi_{\lambda} \otimes \pi_{\mu})\Delta'(x) \cdot R_{\lambda\mu}^{so(2n+1)}, \quad \forall x \in U_{-q}(so(2n+1)),$$

where $\Delta' = P \circ \Delta$ is the opposite co-multiplication. For each such λ , the map $\check{R}_{\lambda\lambda}^{so(2n+1)} = P \circ R_{\lambda\lambda}^{so(2n+1)}$ commutes with the action of $U_{-q}(so(2n+1))$:

$$\check{R}_{\lambda\lambda}^{so(2n+1)} \cdot (\pi_{\lambda} \otimes \pi_{\lambda})\Delta(x) = (\pi_{\lambda} \otimes \pi_{\lambda})\Delta(x) \cdot \check{R}_{\lambda\lambda}^{so(2n+1)}, \quad \forall x \in U_{-q}(so(2n+1)).$$

Similarly, it was shown in [2] that there exist maps $R_{\lambda\mu}^{osp(1|2n)} \in \text{End}_{\mathbb{C}}(V_{\lambda} \otimes V_{\mu})$ and $\check{R}_{\lambda\lambda}^{osp(1|2n)} = P \circ R_{\lambda\lambda}^{osp(1|2n)}$ for all integral dominant weights λ and μ satisfying

$$R_{\lambda\mu}^{osp(1|2n)} \cdot (\bar{\pi}_{\lambda} \otimes \bar{\pi}_{\mu})\Delta(x) = (\bar{\pi}_{\lambda} \otimes \bar{\pi}_{\mu})\Delta'(x) \cdot R_{\lambda\mu}^{osp(1|2n)}, \quad \forall x \in U_q(osp(1|2n)),$$

$$\check{R}_{\lambda\lambda}^{osp(1|2n)} \cdot (\bar{\pi}_{\lambda} \otimes \bar{\pi}_{\lambda})\Delta(x) = (\bar{\pi}_{\lambda} \otimes \bar{\pi}_{\lambda})\Delta(x) \cdot \check{R}_{\lambda\lambda}^{osp(1|2n)}, \quad \forall x \in U_q(osp(1|2n)).$$

We can now define representations of the Braid group B_f in the usual way. Let $\{\sigma_i^{\pm 1} \mid i = 1, \dots, f-1\}$ be the generators of B_f as shown in Fig. 1. Fix $k \geq f$, then for each $i = 1, \dots, k-1$ and each $A \in \{so(2n+1), osp(1|2n)\}$, fix

$$(\check{R}_{\lambda\lambda}^A)^{\pm 1}_i = \text{id}^{\otimes(i-1)} \otimes (\check{R}_{\lambda\lambda}^A)^{\pm 1} \otimes \text{id}^{\otimes(k-i-1)}.$$

The homomorphisms

$$\rho_{\lambda}^A : \sigma_i^{\pm 1} \mapsto (\check{R}_{\lambda\lambda}^A)^{\pm 1}_i$$

then yield representations of B_f from representations of $U_q(osp(1|2n))$ and $U_{-q}(so(2n+1))$.

These representations of B_f , up to a sign, also yield representations of different specialisations of the Birman-Wenzl-Murakami algebra BMW_f , which we will discuss later.

We now detail the Markov traces defined on these representations of the Braid group. For each $A \in \{so(2n+1), osp(1|2n)\}$, let

$$\check{R}_{VV}^{so(2n+1)} \in \text{End}_{U_{-q}(so(2n+1))}(V \otimes V), \quad \text{and} \quad (2.4)$$

$$\check{R}_{VV}^{osp(1|2n)} \in \text{End}_{U_q(osp(1|2n))}(V \otimes V), \quad (2.5)$$

and let \mathcal{C}_f^A be the complex algebra generated by $\left\{ (\check{R}_{VV}^A)^{\pm 1}_i \mid i = 1, 2, \dots, f-1 \right\}$.

Define the maps $\psi^A : \mathcal{C}_f^A \rightarrow \mathbb{C}$ by

$$\psi^{so(2n+1)}(X_1) = \text{tr}_{-q}(X_1) / (\dim_{-q}(V))^f, \quad \forall X_1 \in \mathcal{C}_f^{so(2n+1)}, \quad (2.6)$$

$$\psi^{osp(1|2n)}(X_2) = \text{str}_q(X_2) / (\text{sdim}_q(V))^f, \quad \forall X_2 \in \mathcal{C}_f^{osp(1|2n)}. \quad (2.7)$$

Note that $\dim_{-q}(V) = (-q^{2n} + q^{-2n}) / (q - q^{-1}) + 1$ is the quantum dimension of the $U_{-q}(so(2n+1))$ -module V and that $\text{sdim}_q(V) = \dim_{-q}(V)$.

It is well known that both $\psi^{so(2n+1)}$ and $\psi^{osp(1|2n)}$ are Markov traces and thus can be used to create link invariants following Subsec. 2.1. The resulting link invariants are the uncoloured quantum link invariants obtained by colouring each

component of the link with the $(2n+1)$ -dimensional irreducible representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$, respectively.

In the next section we show how the Kauffman polynomial can be defined using the Birman-Wenzl-Murakami algebra and detail the connections between the Kauffman knot invariant and the quantum link invariants discussed in this section.

3. The Birman-Wenzl-Murakami Algebra and the Kauffman Polynomial

We now discuss the Birman-Wenzl-Murakami algebra [1,3], which admits a naturally defined trace functional from which the Kauffman link invariant can be obtained [6].

3.1. The Birman-Wenzl-Murakami Algebra

Let $f \geq 2$ be an integer. The Birman-Wenzl-Murakami algebra BMW_f is an algebra taken over $\mathbb{C}(r, s)$, the field of rational polynomials in the indeterminates r and s with complex coefficients [6], which is generated by the invertible elements $\{g_i \mid 1 \leq i \leq f-1\}$ that are subject to the relations

$$\begin{aligned} g_i g_j &= g_j g_i, & |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & 1 \leq i \leq f-2, \\ e_i g_i &= r^{-1} e_i, & 1 \leq i \leq f-1, \\ e_i g_{i-1}^{\pm 1} e_i &= r^{\pm 1} e_i, & 1 \leq i \leq f-1, \end{aligned}$$

where e_i is defined by

$$(s - s^{-1})(1 - e_i) = g_i - g_i^{-1}, \quad 1 \leq i \leq f-1.$$

It can be shown that each g_i also satisfies $(g_i - r^{-1})(g_i + s^{-1})(g_i - s) = 0$.

The algebra BMW_f comes naturally equipped with a functional $\text{tr} : BMW_f \rightarrow \mathbb{C}(r, s)$ satisfying ([6, Lem. 3.4 (d)]):

$$\text{tr}(a\chi b) = \text{tr}(\chi)\text{tr}(ab), \quad \forall a, b \in BMW_{f-1}, \quad \chi \in \{g_{f-1}, e_{f-1}\}, \quad (3.8)$$

where we regard each element of BMW_{f-1} as an element of BMW_f under the canonical inclusion (eg $g_i \in BMW_{f-1}$ is regarded as an element of BMW_f via $g_i \mapsto g_i$). We recall the definition of the trace functional tr in Sec. 6 and also the well-known result that it can be used to define the Kauffman link invariant [6, p. 404] (see Sec. 4 for details).

Following [6], in this paper we say that an algebra A is semisimple if it is isomorphic to a direct sum of matrix algebras: $A \cong \bigoplus_i M_{k_i}(\mathbb{C})$, where $k_i \in \{1, 2, \dots\}$ and $M_j(\mathbb{C})$ is the algebra of $j \times j$ matrices with complex entries. The algebra BMW_f is semisimple [6] and its structure can be conveniently represented by a

Bratteli diagram, which is an undirected graph encoding information about a sequence $\mathbb{C} \cong A_0 \subset A_1 \subset A_2 \subset \dots$ of inclusions of finite dimensional semisimple algebras [4].

To write down the Bratteli diagram for BMW_f , we firstly need the Young lattice [6], which is (almost) identical to the Bratteli diagram for the sequence of inclusions of group algebras of the symmetric group: $\mathbb{C}S_1 \subset \mathbb{C}S_2 \subset \mathbb{C}S_3 \subset \dots$. The vertices of the Young lattice are grouped into levels: each Young diagram with $f \geq 0$ boxes labels a vertex on the f^{th} level of the Young lattice, and a vertex λ on the f^{th} level is connected to a vertex μ on the $(f + 1)^{st}$ level by an edge if and only if λ and μ differ by exactly one box. The empty Young diagram (containing no boxes) is on the 0^{th} level.

For each f , let Y_f be the set of vertices on the f^{th} level of the Young lattice and define $\Gamma_f = \bigcup_{f-2k \geq 0} Y_{f-2k}$ where k ranges over all of \mathbb{Z}_+ . The vertices on the

k^{th} level of the Bratteli diagram for BMW_f are precisely the elements of Γ_k and a vertex λ on the k^{th} level is connected to a vertex μ on the $(k + 1)^{st}$ level if and only if λ and μ differ by exactly one box. For clarity, we show the Bratteli diagram for BMW_f up to the 4^{th} level in Fig. 2.

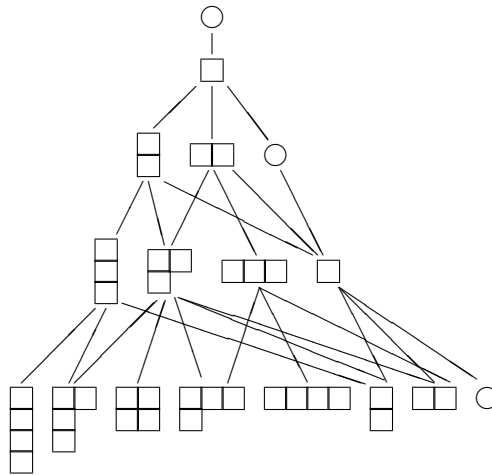


Fig. 2. The Bratteli diagram for BMW_f up to the 4^{th} level inclusive.

We say that R is a path of length f in the Bratteli diagram for BMW_f if $R = (r_0, r_1, \dots, r_f)$ is a sequence of $f + 1$ Young diagrams where $r_k \in \Gamma_k$ for each $k = 0, 1, \dots, f$ and if r_i is connected by an edge to the vertex r_{i+1} in the Bratteli diagram for BMW_f for each $i = 0, 1, \dots, f - 1$. We write $shp(R) = r_f$ and fix Ω_f be the set of pairs (R, S) of paths of length f in the Bratteli diagram for BMW_f

where $\text{shp}(R) = \text{shp}(S)$.

Ram and Wenzl wrote down an explicit basis $\{E_{ST} \in BMW_f \mid (S, T) \in \Omega_f\}$ of BMW_f in [4]. This basis is a set of matrix units, ie the basis elements satisfy $E_{QR}E_{ST} = \delta_{RS}E_{QT}$.

The following fact [6, Lem. 4.2] is important: let R be a path of length f in the Bratteli diagram for BMW_f where $\text{shp}(R) = \lambda$, then $\text{tr}(E_{RR}) = Q_\lambda(r, s)/x^f$, where $Q_\lambda(r, s)$ is the function given in (6.16) and $x = \frac{r - r^{-1}}{s - s^{-1}} + 1$.

3.2. The algebra $BMW_f(t, q)$ with complex parameters t, q

The algebra $BMW_f(t, q)$, obtained by formally replacing the indeterminates r and s in BMW_f with the complex numbers t and q , respectively, is equipped with a functional $\text{tr} : BMW_f(t, q) \rightarrow \mathbb{C}$ satisfying

$$\text{tr}(a\chi b) = \text{tr}(\chi)\text{tr}(ab), \quad \forall a, b \in BMW_{f-1}(t, q), \quad \chi \in \{g_{f-1}, e_{f-1}\}, \quad (3.9)$$

where we regard each element of $BMW_{f-1}(t, q)$ as an element of $BMW_f(t, q)$ under the canonical inclusion.

Define an ideal $J_f(t, q) \subset BMW_f(t, q)$ with respect to tr by

$$J_f(t, q) = \{b \in BMW_f(t, q) \mid \text{tr}(ab) = 0, \forall a \in BMW_f(t, q)\}.$$

If q is nonzero and not a root of unity and $t \neq \pm q^k$ for any $k \in \mathbb{Z}$, then $J_f(t, q) = 0$ and $BMW_f(t, q)$ is semisimple. If q is not a root of unity and $t = \pm q^k$ for some $k \in \mathbb{Z}$, then $J_f(\pm q^k, q)$ may be non-zero and the quotient $BMW_f(\pm q^k, q)/J_f(\pm q^k, q)$ is semisimple (see [6, Cor. 5.6] for details).

In examining the Kauffman link polynomials, we shall use the structures of the semisimple quotients $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ and $BMW_f(q^{2n}, -q)/J_f(q^{2n}, -q)$, which are encoded in truncated Bratteli diagrams. These truncated Bratteli diagrams are in fact identical, which arises directly from the fact that $Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q)$ (Lem. 6.1) and may be related to the fact that $BMW_f(-q^{2n}, q)$ and $BMW_f(q^{2n}, -q)$ are isomorphic algebras [6]. We now explain how the truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ is obtained [6]; the other truncated Bratteli diagram is similarly derived.

Fix q to be generic. Initially, we obtain a subgraph $Y(-q^{2n}, q)$ of the Young lattice inductively. Firstly fix the Young diagram with no boxes to be a vertex in $Y(-q^{2n}, q)$. The inductive step is as follows: assume that the Young diagram λ is a vertex in $Y(-q^{2n}, q)$ and that the Young diagram μ differs from λ by exactly one box. Then μ is also a vertex in $Y(-q^{2n}, q)$ if the function $Q_\mu(-q^{2n}, q)$ is non-zero, where $Q_\mu(-q^{2n}, q)$ is given in (6.16). Explicitly from [2,6], a Young diagram λ is a vertex in $Y(-q^{2n}, q)$ if and only if $\lambda'_1 + \lambda'_2 \leq 2n + 1$, where λ'_i is the number of boxes in the i^{th} column of λ from the left.

The truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ is then the subgraph of the Bratteli diagram for BMW_f obtained by removing all vertices

that do not belong to $Y(-q^{2n}, q)$. We show the truncated Bratteli diagram for $BMW_f(-q^2, q)/J_f(-q^2, q)$ up to the 4th level in Fig. 3.

The truncated Bratteli diagram for $BMW_f(q^{2n}, -q)/J_f(q^{2n}, -q)$ is obtained in the same way as is the truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ except that we replace $(-q^{2n}, q)$ with $(q^{2n}, -q)$. The fact that $Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q)$ means that the truncated Bratteli diagrams for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ and $BMW_f(q^{2n}, -q)/J_f(q^{2n}, -q)$ are identical.

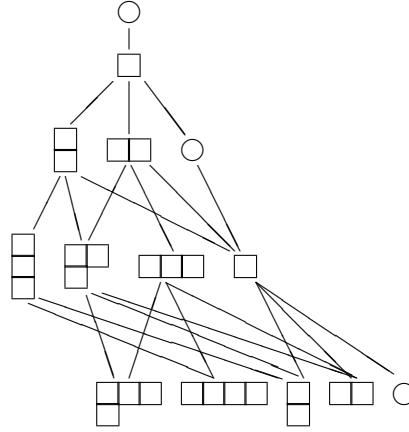


Fig. 3. The Bratteli diagram for $BMW_f(-q^2, q)/J_t(-q^2, q)$ up to the 4th level inclusive.

Note that $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q) \cong \bigoplus_\lambda M_{b(\lambda)}(\mathbb{C})$ where the direct sum is over all vertices λ on the f^{th} level of the truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ and $b(\lambda)$ is the number of paths of length f in the truncated Bratteli diagram ending at λ .

We are almost at the point where we can write down a basis for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$. Fix $\Omega_f(-q^{2n}, q)$ to be the set of pairs (R, S) of paths of length f in the truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ where $shp(R) = shp(S)$. The matrix units

$$\{E_{RS}(-q^{2n}, q) \in BMW_f(-q^{2n}, q) \mid (R, S) \in \Omega_f(-q^{2n}, q)\},$$

obtained by replacing the indeterminates r and s with the complex numbers $-q^{2n}$ and q , respectively, in a subset of the matrix units of BMW_f , are all well-defined and non-zero [2]. It is very important to note that $\text{tr}(E_{SS}(-q^{2n}, q)) \neq 0$ for all $(S, S) \in \Omega_f(-q^{2n}, q)$ and that $E_{RS}(-q^{2n}, q) \notin J_f(-q^{2n}, q)$ for all $(R, S) \in \Omega_f(-q^{2n}, q)$.

3.3. Representations of $BMW_f(-q^{2n}, q)$ and $BMW_f(q^{2n}, -q)$

Certain representations of $U_q(\mathfrak{sp}(2n))$, $U_q(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$ yield representations of different specialisations of $BMW_f(t, q)$. In this subsection, we recall how representations of $U_q(\mathfrak{osp}(1|2n))$ yield representations of $BMW_f(-q^{2n}, q)$ [2] and how representations of $U_{-q}(\mathfrak{so}(2n+1))$ yield representations of $BMW_f(q^{2n}, -q)$ [5].

Fix q to be generic and non-zero in this rest of this section and in Secs. 4 and 5. Recall the maps $\check{R}_{VV}^{\mathfrak{so}(2n+1)}$ and $\check{R}_{VV}^{\mathfrak{osp}(1|2n)}$ from (2.4) and (2.5), respectively. The homomorphism

$$\rho_V^{\mathfrak{so}(2n+1)} : g_i^{\pm 1} \mapsto \left(\check{R}_{VV}^{\mathfrak{so}(2n+1)} \right)_i^{\pm 1}$$

yields a representation of $BMW_f(q^{2n}, -q)$ [5], and the homomorphism

$$\Upsilon : g_i^{\pm 1} \mapsto - \left(\check{R}_{VV}^{\mathfrak{osp}(1|2n)} \right)_i^{\pm 1}$$

yields a representation of $BMW_f(-q^{2n}, q)$ [2].

Recall the definitions of the Markov traces $\psi^{\mathfrak{so}(2n+1)}$ and $\psi^{\mathfrak{osp}(1|2n)}$ from (2.6) and (2.7), respectively, and the trace functional tr on the Birman-Wenzl-Murakami algebra. It can be shown that

$$\psi^{\mathfrak{so}(2n+1)} \left(\rho_V^{\mathfrak{so}(2n+1)}(a) \right) = \text{tr}(a), \quad \forall a \in BMW_f(q^{2n}, -q), \quad (3.10)$$

$$\psi^{\mathfrak{osp}(1|2n)}(\Upsilon(a)) = \text{tr}(a), \quad \forall a \in BMW_f(-q^{2n}, q), \quad (3.11)$$

where Eq. (3.11) is proved in [2]. This equality of the traces allows one to use the traces of elements of $BMW_f(\pm q^{2n}, \mp q)$ in calculating a link polynomial following Eq. (2.1).

In the next section we recall how the Kauffman link invariant can be obtained using tr . Eqs. (3.10) and (3.11) will then give the essential connections between the uncoloured $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ quantum link invariants and the Kauffman link invariant defined directly from the Birman-Wenzl-Murakami algebra.

4. Link Invariants from the Birman-Wenzl-Murakami Algebra

The Kauffman link invariant can be defined using BMW_f as follows [6]. Let a link L be presented as a braid on f strings with corresponding braid group element $b = \sigma_{i_1}^{m_1} \sigma_{i_2}^{m_2} \cdots \sigma_{i_j}^{m_j} \in B_f$, where $m_k \in \mathbb{Z}$ for each k , and let $\beta \in BMW_f$ be the image of b under the homomorphism $\sigma_i^{\pm 1} \mapsto g_i^{\pm 1}$. Define $e(b) = \sum_k m_k$ and $\widehat{\text{tr}}(b) = r^{-e(b)} \text{tr}(\beta)$, then the Kauffman link invariant of L is

$$F_L(r, s) = \widehat{\text{tr}}(\sigma_1)^{1-f} \widehat{\text{tr}}(b). \quad (4.12)$$

The image $\beta(-q^{2n}, q)$ of b under the homomorphism $\sigma_i^{\pm 1} \mapsto g_i^{\pm 1} \in BMW_f(-q^{2n}, q)$ is well-defined, as is the image $\beta(q^{2n}, -q)$ of b under the homomorphism $\sigma_i^{\pm 1} \mapsto g_i^{\pm 1} \in BMW_f(q^{2n}, -q)$.

As an aside, let us examine the link invariant $\tilde{F}(L)$ from Eq. (2.1). The homomorphism $\rho : \sigma_i^{\pm 1} \mapsto g_i^{\pm 1} \in BMW_f$ yields a representation of B_f , and $\text{tr} \circ \rho$ is a functional satisfying the three Markov properties:

- (i) $\text{tr}(ab) = \text{tr}(ba)$, $\forall a, b \in BMW_f$,
- (ii) $\text{tr}(ag_{f-1}) = \text{tr}(g_{f-1})\text{tr}(a) = \frac{r}{x}\text{tr}(a)$, $\forall a \in BMW_{f-1}$,
- (iii) $\text{tr}(ag_{f-1}^{-1}) = \text{tr}(g_{f-1}^{-1})\text{tr}(a) = \frac{r^{-1}}{x}\text{tr}(a)$, $\forall a \in BMW_{f-1}$.

It follows that $\text{tr} \circ \rho$ can be used to construct the link invariant $\tilde{F}(L)$. Given a link L presented as the closure of a braid with corresponding braid group element $b \in B_f$, one immediately obtains

$$\tilde{F}(L) = x^{f-1} r^{-e(b)} \text{tr}(\rho(b)),$$

from Eq. (2.1), which is evidently equal to the right hand side of Eq. (4.12) upon applying simple substitutions to the latter.

It immediately follows that the quantum link invariant $\tilde{F}(L)$ obtained by colouring each component of the link L with the $(2n+1)$ -dimensional irreducible representation of $U_q(\mathfrak{osp}(1|2n))$ (resp. $U_{-q}(\mathfrak{so}(2n+1))$) is *identical* to the Kauffman link invariant $F_L(-q^{2n}, q)$ (resp. $F_L(q^{2n}, -q)$) from Eq. (4.12).

5. The relationship between $F_L(-q^{2n}, q)$ and $F_L(q^{2n}, -q)$

We shall now examine the relationship between $F_L(-q^{2n}, q)$ and $F_L(q^{2n}, -q)$ and show that, in principle, each Kauffman link invariant can be obtained from the other by a judicious substitution of variables, although the precise substitution may be obscure. It immediately follows from the previous section that a similar result holds for the uncoloured $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ quantum link invariants.

Recall that Ω_f is the set of pairs (R, S) of paths of length f in the Bratteli diagram for BMW_f where $\text{shp}(R) = \text{shp}(S)$. Writing each element $X \in BMW_f$ as a linear combination of matrix units: $X = \sum_{(S,T) \in \Omega_f} c_{ST} E_{ST}$, $c_{ST} \in \mathbb{C}(r, s)$, the trace of X is

$$\text{tr}(X) = \sum_{(S,S) \in \Omega_f} c_{SS} \text{tr}(E_{SS}),$$

where $\text{tr}(E_{SS}) \neq 0$ for all $(S, S) \in \Omega_f$. Given such an element X , we fix

$$X(t, q) = \sum_{(S,T) \in \Omega_f} c_{ST}(t, q) E_{ST}(t, q)$$

to be the corresponding element of $BMW_f(t, q)$ obtained by replacing the indeterminates r and s in X with the complex numbers t and q , respectively.

Recall that $\Omega_f(-q^{2n}, q)$ is the set of pairs (R, S) of paths of length f in the truncated Bratteli diagram for $BMW_f(-q^{2n}, q)/J_f(-q^{2n}, q)$ where $\text{shp}(R) = \text{shp}(S)$ and that $\Omega_f(q^{2n}, -q)$ is similarly defined.

Let $X \in BMW_f$ be any element where each of $X(-q^{2n}, q)$ and $X(q^{2n}, -q)$ is well-defined. Then

$$\begin{aligned} \mathrm{tr}(X)|_{(r,s)=(-q^{2n},q)} &= \mathrm{tr}(X(-q^{2n}, q)) = \sum_{(S,T) \in \Omega_f(-q^{2n},q)} c_{ST}(-q^{2n}, q) \mathrm{tr}(E_{ST}(-q^{2n}, q)) \\ &= \left(\sum_{(S,T) \in \Omega_f(-q^{2n},q)} c_{ST} \mathrm{tr}(E_{ST}) \right) \Big|_{(r,s)=(-q^{2n},q)} \end{aligned} \quad (5.13)$$

as $\mathrm{tr}(E_{SS})|_{(r,s)=(-q^{2n},q)} = 0$ if $(S, S) \notin \Omega_f(-q^{2n}, q)$, and similarly

$$\mathrm{tr}(X)|_{(r,s)=(q^{2n},-q)} = \mathrm{tr}(X(q^{2n}, -q)) = \left(\sum_{(S,T) \in \Omega_f(q^{2n},-q)} c_{ST} \mathrm{tr}(E_{ST}) \right) \Big|_{(r,s)=(q^{2n},-q)}. \quad (5.14)$$

Note that the sums on the right hand sides of (5.13) and (5.14) are over the same sets; the equality of the sets $\Omega_f(-q^{2n}, q)$ and $\Omega_f(q^{2n}, -q)$ arises from the result in Lem. 6.1 that $Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q)$.

Let us rewrite parts of Eqs. (5.13) and (5.14):

$$\left(\sum_{(S,T) \in \Omega_f(-q^{2n},q)} c_{ST} \mathrm{tr}(E_{ST}) \right) \Big|_{(r,s)} = \begin{cases} \mathrm{tr}(X(-q^{2n}, q)), & \text{if } (r, s) = (-q^{2n}, q), \\ \mathrm{tr}(X(q^{2n}, -q)), & \text{if } (r, s) = (q^{2n}, -q). \end{cases} \quad (5.15)$$

It follows from (5.15) that it is in principle possible to obtain $\mathrm{tr}(X(q^{2n}, -q))$ by substituting the pair $(q^{2n}, -q)$ for the pair $(-q^{2n}, q)$ in $\mathrm{tr}(X(-q^{2n}, q))$ (and similarly possible to obtain $\mathrm{tr}(X(-q^{2n}, q))$ from $\mathrm{tr}(X(q^{2n}, -q))$), however, it may be difficult to do this in practise as any appearance of $-q^{2n}$ in $\mathrm{tr}(X(-q^{2n}, q))$ is either the $-q^{2n}$ appearing in the pair to be substituted, or it is a power of the q in the pair to be substituted up to a sign, and these are mapped to different numbers under the substitution.

If the left hand side of Eq. (5.15) is explicitly known, however, this problem immediately disappears.

Considering the definition of the Kauffman link invariant in Eq. (4.12) in light of this discussion, the uncoloured quantum link invariants arising from $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ have a symmetry given by substituting the relevant pairs of signed powers of q . However, it may not be possible to directly obtain one of the quantum link invariants from the other by applying the relevant substitutions without additional knowledge concerning the trace of certain elements in the un-specialised Birman-Wenzl-Murakami algebra.

Similar results will hold for any Kauffman link invariants $F_L(r, s)$ and $F_L(r', s')$ where the truncated Bratteli diagrams for the relevant semisimple quotients of $BMW_f(r, s)$ and $BMW_f(r', s')$ are identical.

5.1. The case for q a root of unity

In this paper, we have not considered the relationship between the relevant quantum link invariants (nor Kauffman link invariants) when q is a root of unity, however, we believe that similar results may hold at roots of unity as at generic q . For q a root of unity, the truncated Bratteli diagram for $BMW_f(\mp q^{2n}, \pm q)/J_f(\mp q^{2n}, \pm q)$ is, for sufficiently large f (depending on the root of unity), a proper subgraph of the truncated Bratteli diagram for $BMW_f(\mp q^{2n}, \pm q)/J_f(\mp q^{2n}, \pm q)$ when q is generic [2,6]. The fact that this subgraph is proper is intimately related to the existences of the truncated dominant Weyl alcoves in the relevant weight spaces of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ when q is a root of unity.

6. Technical results

6.1. Definition of tr

For each element $a \in BMW_{f+1}$, there exists a unique element $\epsilon_f(a) \in BMW_f$ such that $e_{f+1}ae_{f+1} = x\epsilon_f(a)e_{f+1}$ where $\epsilon_f : BMW_{f+1} \rightarrow BMW_f$ is a linear map. The tr functional on BMW_f is then inductively defined by $\text{tr}(1) = 1$ and $\text{tr}(a) = \text{tr}(\epsilon_f(a))$ for $a \in BMW_{f+1}$. In particular, $\text{tr}(e_i) = 1/x$ and $\text{tr}(g_i^{\pm 1}) = r^{\pm 1}/x$ for all i .

6.2. Definition of $Q_\lambda(r, s)$

Let λ be a Young diagram, let (i, j) denote the box in the i^{th} row and the j^{th} column of λ and let λ_i (resp. λ'_j) denote the number of boxes in the i^{th} row (resp. j^{th} column) of λ . Denote the Young diagram λ by $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ where the i^{th} row of the Young diagram contains λ_i boxes for each $i = 1, 2, \dots, k$, and the l^{th} row contains no boxes for each $l > k$. The function $Q_\lambda(r, s)$ is

$$Q_\lambda(r, s) = \prod_{(j,j) \in \lambda} \frac{rs^{\lambda_j - \lambda'_j} - r^{-1}s^{-\lambda_j + \lambda'_j} + s^{\lambda_j + \lambda'_j - 2j + 1} - s^{-\lambda_j - \lambda'_j + 2j - 1}}{s^{h(j,j)} - s^{-h(j,j)}} \times \prod_{(i,j) \in \lambda, i \neq j} \frac{rs^{d(i,j)} - r^{-1}s^{-d(i,j)}}{s^{h(i,j)} - s^{-h(i,j)}}, \quad (6.16)$$

where the hooklength $h(i, j)$ is defined by $h(i, j) = \lambda_i - i + \lambda'_j - j + 1$, and where

$$d(i, j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1, & \text{if } i \leq j, \\ -\lambda'_i - \lambda'_j + i + j - 1, & \text{if } i > j. \end{cases}$$

Intuitively, the hooklength $h(i, j)$ is the number of boxes in the hook cornered on the box (i, j) , ie the number of boxes below the (i, j) box in the j^{th} column plus the number of boxes to the right of the (i, j) box in the i^{th} row, plus one.

6.3. Technical lemma

Lemma 6.1. *For each Young diagram λ ,*

$$Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q). \quad (6.17)$$

Proof. Simple calculations show that (6.17) is true if and only if

$$\prod_{\substack{(i,j) \in \lambda \\ i \neq j}} -q^{2n+d(i,j)} + q^{-2n-d(i,j)} = \left(\prod_{\substack{(i,j) \in \lambda \\ i < j}} (-1)^{\lambda'_j + \lambda_j} (q^{2n+d(i,j)} - q^{-2n-d(i,j)}) \right) \\ \times \left(\prod_{\substack{(i,j) \in \lambda \\ i > j}} (-1)^{\lambda_i + \lambda'_i} (q^{2n+d(i,j)} - q^{-2n-d(i,j)}) \right)$$

and this last equation is true if

$$\prod_{\substack{(i,j) \in \lambda \\ i \neq j}} (-1) = \left(\prod_{\substack{(i,j) \in \lambda \\ i < j}} (-1)^{\lambda'_j + \lambda_j} \right) \left(\prod_{\substack{(i,j) \in \lambda \\ i > j}} (-1)^{\lambda_i + \lambda'_i} \right). \quad (6.18)$$

We now show that (6.18) is true. Define the following sets:

$$\text{Hor}_k = \{(k, j) \in \lambda \mid j = 1, 2, \dots, \min\{k-1, \lambda_k\}\} \\ \text{Ver}_k = \{(i, k) \in \lambda \mid i = 1, 2, \dots, \min\{k-1, \lambda'_k\}\}.$$

Noting that $|\text{Ver}_k \cap \text{Hor}_l| = 0$ for all k and l and that $|\text{Hor}_k \cap \text{Hor}_l| = 0 = |\text{Ver}_k \cap \text{Ver}_l|$ for all $k \neq l$, it follows that (6.18) is true if the following equation holds for each k :

$$(-1)^{|\text{Hor}_k \cup \text{Ver}_k|} = \left(\prod_{(i,k) \in \text{Ver}_k} (-1)^{\lambda'_k + \lambda_k} \right) \left(\prod_{(k,j) \in \text{Hor}_k} (-1)^{\lambda_k + \lambda'_k} \right). \quad (6.19)$$

If $|\text{Hor}_k \cup \text{Ver}_k|$ is even, the right hand side of (6.19) clearly equals 1 as Hor_k and Ver_k are disjoint. Alternatively, if $|\text{Hor}_k \cup \text{Ver}_k|$ is odd, then $\lambda_k \leq k-2$ and/or $\lambda'_k \leq k-2$. If $\lambda'_k \leq k-2$, then $\lambda_k \leq k-1$ as λ is a Young diagram. Similarly, if $\lambda_k \leq k-2$, then $\lambda'_k \leq k-1$ as λ is a Young diagram. In both cases it follows that $\lambda'_k = |\text{Ver}_k|$ and $\lambda_k = |\text{Hor}_k|$. If $|\text{Hor}_k \cup \text{Ver}_k|$ is odd, $\lambda_k + \lambda'_k$ is also odd as $\lambda_k + \lambda'_k = |\text{Hor}_k| + |\text{Ver}_k| = |\text{Hor}_k \cup \text{Ver}_k|$, and clearly the right hand side of (6.19) equals -1 . Thus (6.19) is true for each k , from which it follows that (6.18) is true, which completes the proof of the lemma. \square

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